MATH-UA 9263 - Partial Differential Equations
Recitation 1: general intro + diffusion

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Question 1 (Vasy, Evans) *Classify the following PDEs by degree of non-linearity (linear,semilinear, quasilinear, fully non linear)*

(i) $(\cos x)u_x + u_y = u^2$

(ii) $uu_{tt} = u_{xx}$

(iii) $u_x - e^x u_y = \cos x$

(iv) $u_{tt} - u_{xx} + e^u u_x = 0$

(v) $|Du| = 1$

(vi) $\operatorname{div}\left(|Du|^{p-2}Du\right) = 0$

(vii) $u_t - \Delta(u^\gamma) = 0$

(viii) $u_t - \sum_{ij=1}^{n} \left(a^{ij}u\right)_{x_i,x_j} - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$

(ix) $u_t - \sum_{i=1}^{n} \left(b^i u\right)_{x_i} = 0$

(x) $\operatorname{div}\left(\frac{Du}{\left(1+|Du|^2\right)^{1/2}}\right) = 0$

Question 2 (Vasy 2.1) *Recall that if* $\Omega \subset \mathbb{R}^3$ *is a domain and* $u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3$ *is* $\mathcal{C}^1$*, the the divergence of* u *is the function* $\nabla \cdot u : \Omega \to \mathbb{R}$

$$\nabla \cdot u = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3,$$

While the curl of u *is the vector field* $\nabla \times u : \Omega \to \mathbb{R}^3$,

$$\nabla \times u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)$$

Further, for $\mathcal{C}^2$ *functions* u, $\nabla^2 u : \Omega \to \mathbb{R}^3$ *is the vector field*

$$\begin{aligned}\nabla^2 u &= \left(\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3\right)\\ &= \left((\partial_1^2 + \partial_2^2 + \partial_3^2)u_1, (\partial_1^2 + \partial_2^2 + \partial_3^2)u_2, \left(\partial_1^2 + \partial_2^2 + \partial_3^2\right)u_3\right)\end{aligned}$$

Show that

$$\nabla \times (\nabla \times u) = \nabla(\nabla \cdot u) - \nabla^2 u$$

Question 3 *Find the distribution of temperature in a long cylindrical tube with inner radius* $r = a$ *and outer radius* $r = b$ *which is maintained at temperature* $u(r = a) = T_a$ *and* $u(r = b) = T_b$*. (Stationnary case without heat generation)*

Question 4 *We consider the heat equation in cylindrical coordinates*

$$\frac{\partial u}{\partial t} = \frac{k_0}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad a < r < b \tag{1}$$

(i) *Derive the equation above from the Laplacian and the change of variables*

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

(ii) *We consider equation* (1) *subject to the following conditions*

$$u(r,0) = f(r),\ \frac{\partial u}{\partial r}(a,t) = \beta,\ \frac{\partial u}{\partial r}(b,t) = 1$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

Question 5 *Find the temperature profile (steady state) in a long and thin wall with the following parameters: $u(x=0) = 1000°C = T_1$, $u(x=L) = 200°C = T_2$, $L = 1m$, $k_0 = 1W/m°C$. Same question if the conductivity varies as $k_0 + k_1 u$.*

Question 6 *Consider a uniform circular disk whose entire surface is insulated. Assume that its temperature at $t = 0$ is a function only of the distance r from the center of the disk. Starting from the basic physical laws, show that the temperature $u(r,t)$ of the disk satisfies the equation*

$$\frac{\partial u}{\partial t} = K\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{2}$$

Question 7 *Consider the flow of heat in a long thin bar (one dimensional case). If the bar gains heat due to radioactive decay, determine the equation satisfied by the temperature $u(x,t)$. Assume that the rate at which heat is gained per unit volume by radioactive decay is proportional to e^{-ax}.*

Question 8 *A thin bar is conducting heat along its length and is also radiating heat from its surface. The temperature of the surroundings is U. Assume that the radiation obeys Newton's law, that is, the time rate of change of temperature due to radiation is proportional to the temperature difference between the body and its surroundings. The bar has cross-sectional area A and a density δ. Derive the differential equation satisfied by the temperature.*

Question 9 *Consider the one dimensional heat equation in the case where heat is gained by the decomposition of the material in the bar. Derive the differential equation satisfied by the temperature if the rate at which heat is gained per unit volume by decomposition is a constant, A.*

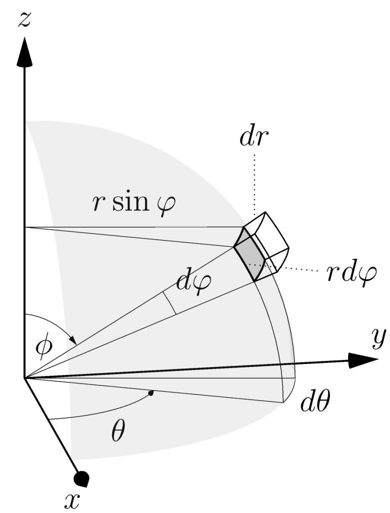

Figure 1: Spherical surface and volume elements

Question 10 *Assume that the temperature is spherically symmetric,* $u = u(r, t)$*, where* r *is the distance from a fixed point (*$r^2 = x^2 + y^2 + z^2$*). Consider the heat flow (without sources) between any two concentric spheres or radii* a *and* b*.*

1. *Show that the total heat energy is* $4\pi \int_a^b c\rho u r^2 \, dr$
2. *Show that the flow of heat energy per unit time out of the spherical shell at* $r = b$ *is* $-4\pi b^2 K_0 \partial u / \partial r \big|_{r=b}$*. A similar result holds at* $r = a$*.*
3. *Use parts* (i) and (ii) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$$

Question 11 *Determine the steady state temperature distribution between two concentric spheres of radii* 1 *and* 4 *respectively, if the outer sphere is maintained at* 80° *and the inner sphere is maintained at* 0°

References

[1] Jean-François Remacle, Grégoire Winckelmans, *FSAB1103 - Mathématiques 3/Équations aux Dérivées Partielles*, 2007.

[2] Richard Haberman, *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Fourth Edition, Pearson 2004.