Introduction to optimisation

Recitation 01b
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Question 1 *Show that a polytope defined by a finite number of linear inequalities is a convex set.*

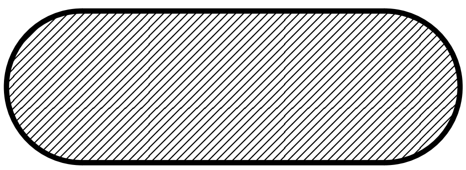
Question 2 *Show that the disk* $\left\{(x,y) \mid x^2+y^2 \leq r^2\right\}$ *is a convex set. How about the* n*-dimensional* ℓ_2 *ball* $\left\{\boldsymbol{x} \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq R^2\right\}$

Question 3 *Extend the previous question by showing that every ball* $B(\boldsymbol{a}, r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\}$ *where* $\boldsymbol{a} \in \mathbb{R}^n$ *and* $r \geq 0$ *is convex.*
Hint: you might want to use the fact that a norm, $\|\cdot\|$*, is a real valued function that satisfies the following properties:*

- *Subadditivity/Triangle inequality:* $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$
- *Absolute homogeneity* $\|\alpha\boldsymbol{x}\| = |\alpha|\|\boldsymbol{x}\|$ *for any scalar* α
- *Positive definiteness: if* $\|\boldsymbol{x}\| = 0$ *then* $\boldsymbol{x} = 0$

Question 4 *Let* $S = \left\{(x,y,z) \mid z \geq x^2+y^2\right\} \subset \mathbb{R}^3$*. Sketch the set and verify that it is a convex set. What are the extreme points and vertices of* S*?*

Question 5 *Is the set defined below convex?*



If so, indicate the extreme points and vertices of the set. Are there extreme points that are not vertices? if yes, indicate those on the set.

Question 6 *Find the convex hull of the set* $\left\{(x, y) \in \mathbb{R}^2 \mid |xy| \leq 1\right\}$

Question 7 *Consider the point sets given in Fig. 1. For each set, draw the convex hull of the set.*

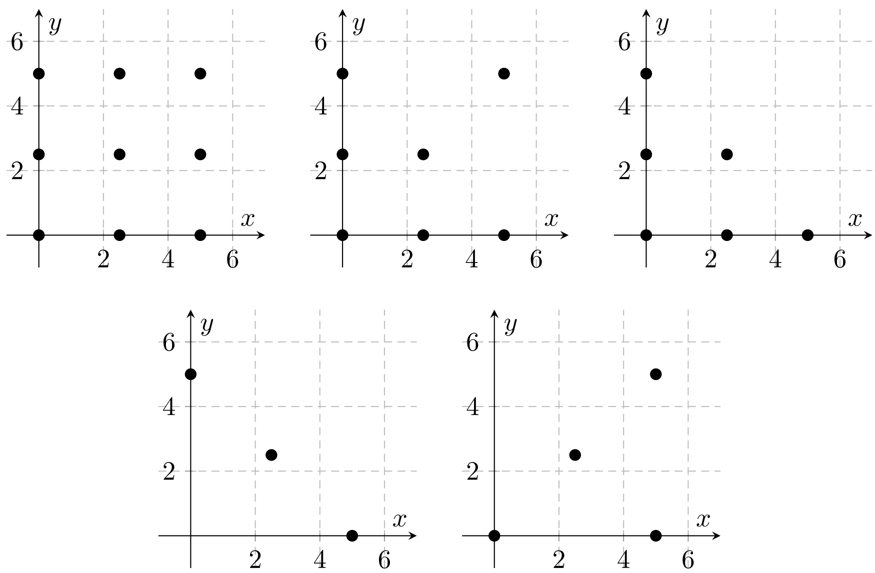


Figure 1: Feasible sets used in Question 7.

Question 8 ([2]) *Let* $c_i \geq 0$ *be non negative constants. Write the following problem as a linear program:*

$$\begin{aligned} \min_{\boldsymbol{x}} \sum_{i=1}^{n} c_i |x_i| \\ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned} \tag{1}$$

Question 9 ([2]) *Another approach when considering absolute values is to introduce the decomposition* $x_i = x_i^+ + x_i^-$. *That is to say, if we are given the problem*

$$\begin{aligned} \min \sum_{i=1}^{n} c_i |x_i| \\ \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned} \tag{2}$$

One can rewrite this problem as

$$\begin{aligned} &\min \sum_{i=1}^{n} c_i(x_i^+ + x_i^-) \\ &\boldsymbol{A}\boldsymbol{x}^+ - \boldsymbol{A}\boldsymbol{x}^- \geq \boldsymbol{b} \\ &x_i^+, x_i^- \geq 0 \end{aligned} \tag{3}$$

where $\boldsymbol{x}^+ = [x_1^+, \ldots, x_n^+]$ *and* $\boldsymbol{x}^- = [x_1^-, \ldots, x_n^-]$. *Formulation* (3) *only makes sense if either* x_i^+ *or* x_i^- *is equal to zero (can you see why?). Show that for the linear program* (3), *the optimal solution must always satisfy* $x_i^+ = 0$ *or* $x_i^- = 0$ *for every* i.

Question 10 ([2]) *In learning, we are often given* n *data pairs* $\{\boldsymbol{x}_i, t_i\}$ *where* $\boldsymbol{x}_i \in \mathbb{R}^n$ *and we are interested in building a linear model of the form* $\boldsymbol{\beta}^T \boldsymbol{x}_i = t_i$ *where* $\boldsymbol{\beta}$ *is the vector of parameters to be estimated. One possible approach consists in solving the following problem*

$$\min_{\boldsymbol{\beta}} \max_{i} \left| t_i - \boldsymbol{x}_i^T \boldsymbol{\beta} \right| \tag{4}$$

Write the linear formulation for this problem.

Question 11 *A non negative function* f *is called log-convex if* $\log(f)$ *is itself convex. Show that the function* $e^{|x|^p}$ *is log-convex when* $p \geq 1$.

Question 12 ([3]) *Let* $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_K \in \mathbb{R}^n$ *be distinct. Consider the set of points that are close to* $\boldsymbol{x}_k$ *than to the other* $\boldsymbol{x}_i$:

$$V = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{x}_k\|_2 \leq \|\boldsymbol{x} - \boldsymbol{x}_i\|_2, i \neq k\}$$

Show that V *is a polyhedron. Express* V *in the form* $V = \{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$

Question 13 *Consider the problem*

$$\min_{\boldsymbol{x}} \quad 2x_1 + 3|x_2 - 9| \tag{5}$$

$$s.t. \quad |x_1| + |x_2 + 3| \leq 6 \tag{6}$$

Write this problem as a linear program.

Question 14 *Let* C_1 *and* C_2 *be two convex sets. Show that the set*

$$C_1 + C_2 = \{\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2 | \boldsymbol{x}_1 \in C_1, \boldsymbol{x}_2 \in C_2\} \tag{7}$$

is convex

Question 15 *Consider the following polyhedron in standard form*

$$P = \left\{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\right\}, \quad \text{where } \boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{8}$$

How many extreme points does P *have?*

Question 16 ([4]) *A collection of points* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m \in \mathbb{R}^n$ *are called* affinely independent *if for* $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$*, with* $\sum_{i=1}^m \lambda_i = 0$*, it follows that whenever* $\sum_{i=1}^m \lambda_i \boldsymbol{x}_i = 0$*, we must have* $\lambda_1 = \ldots = \lambda_m = 0$*. Assume that* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in \mathbb{R}^n$ *are such that each* $\boldsymbol{x} \in conv\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ *is a* unique *convex combination of the* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$*. Show that* $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ *are affinely independent.*

Question 17 ([4]) *For a set* $A \subset \mathbb{R}^n$*, the polar of* A*,* A° *can be defined as*

$$A^\circ = \{\boldsymbol{x} \in \mathbb{R}^n | \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1, \text{ for all } y \in A\} \tag{9}$$

Show that A° *is convex. Then show that if* P *is a polytope then* P° *is a polyhedral set.*

References

[1] Pablo Pedregal, *Introduction to optimization.* Vol. 46. New York: Springer, 2004.

[2] Bertsimas, Dimitris and Tsitsiklis, John N, *Introduction to linear optimization*, Vol. 6, Athena scientific Belmont, MA, 1997.

[3] Boyd, Stephen P and Vandenberghe, Lieven, *Convex optimization*, Cambridge university press, 2004.

[4] , Daniel Hug, Wolfgang Weil, *Lectures on convex geometry*, Springer 2020.